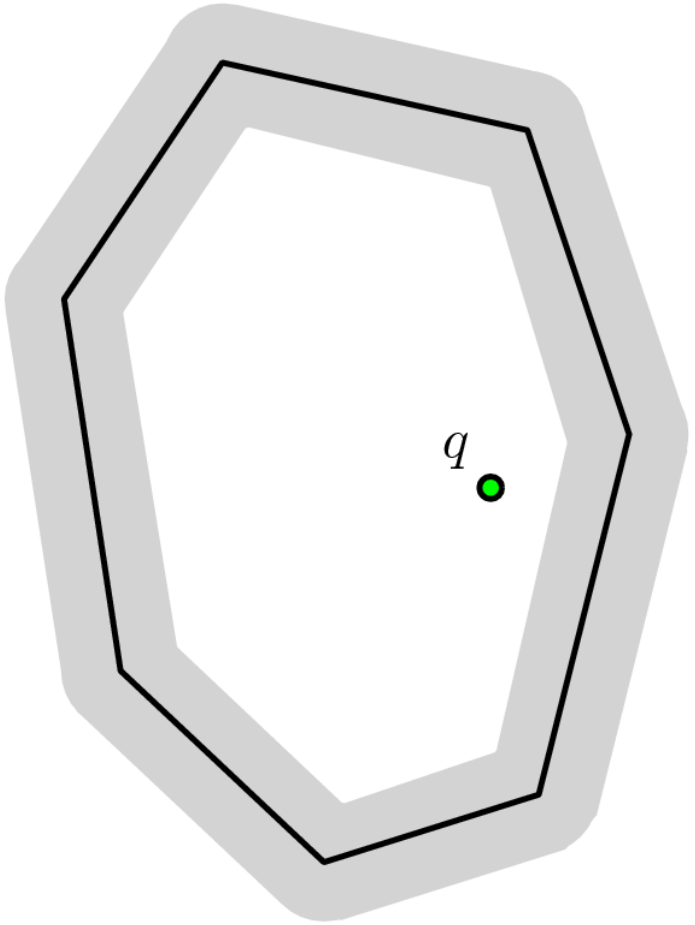
Approximate Polytope Membership Queries
(to appear in STOC'11)



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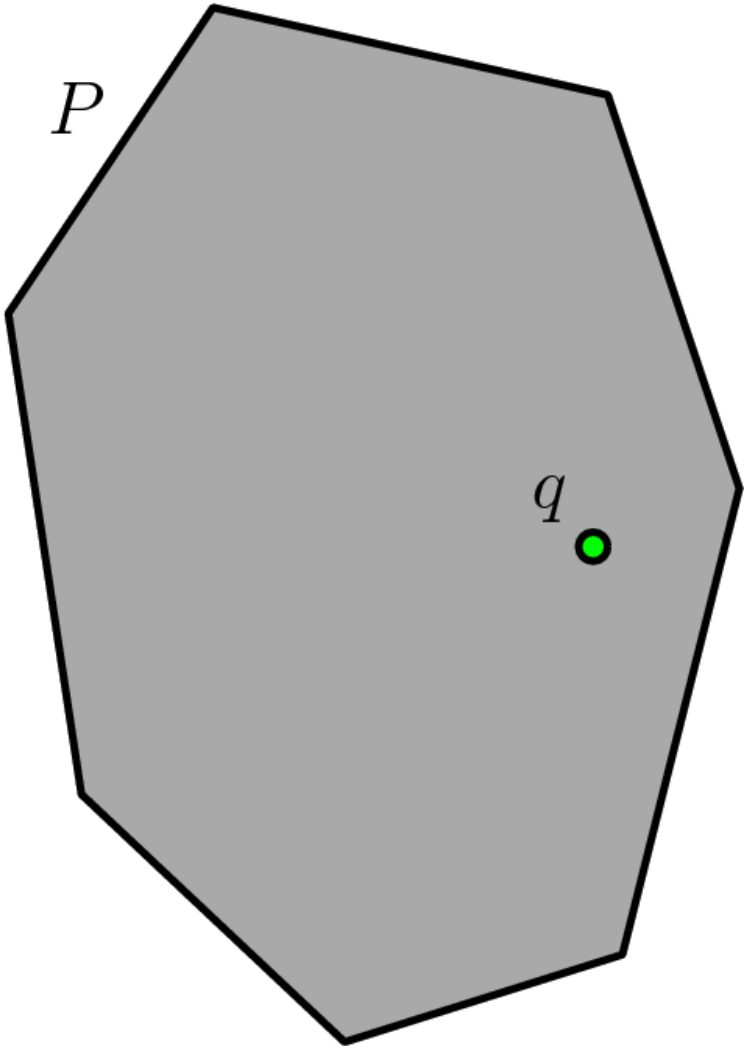
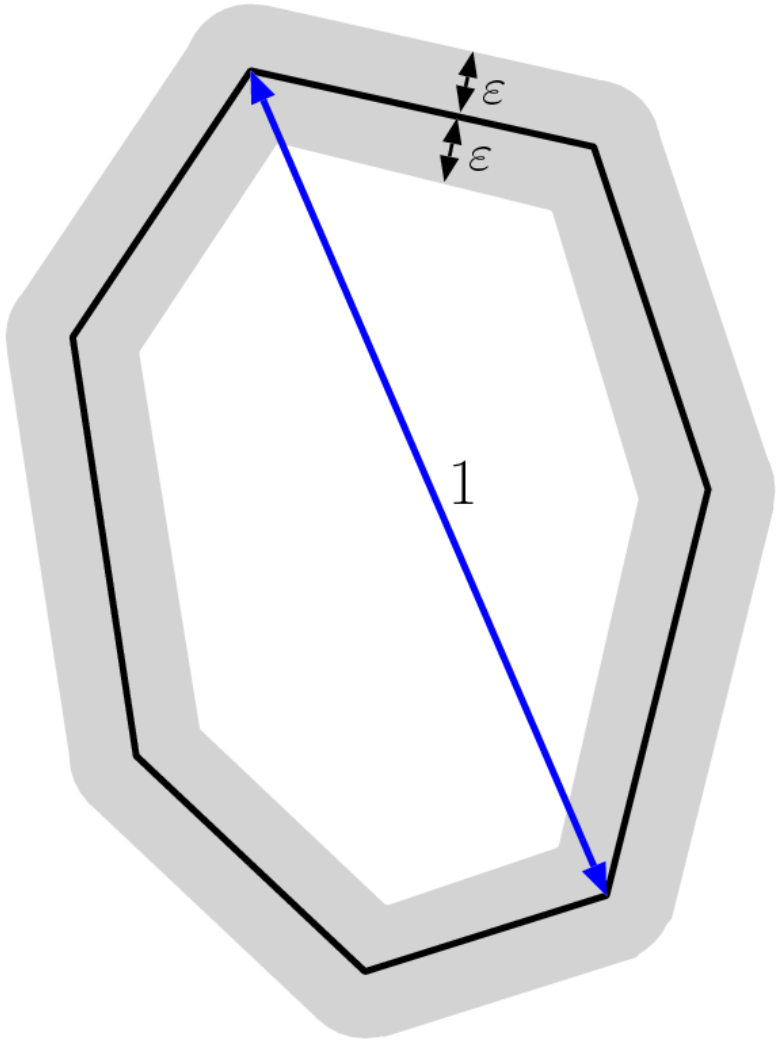
Polytope Membership Queries



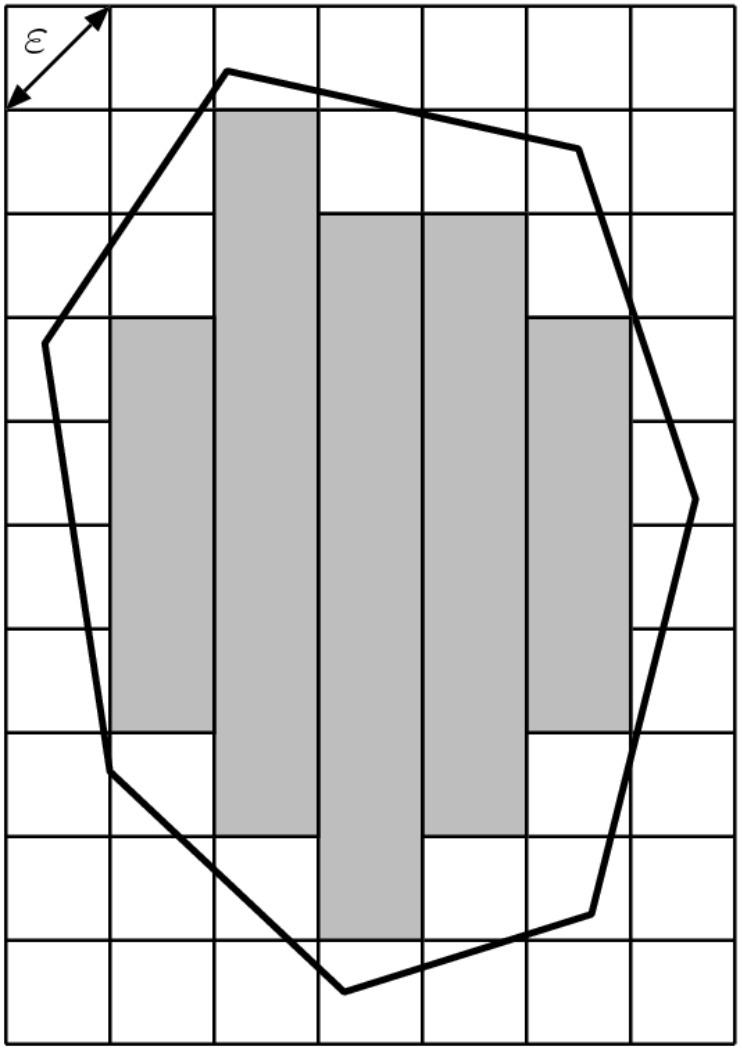
- Data: Polytope *P* in *d*-dimensional space, given as intersection of *n* halfspaces
- Query: Given a point *q*, is *q* inside *P*?
- For d ≤ 3, it can be solved solved with storage O(*n*) and query time O(log *n*)
- Dual of halfspace emptiness searching

Approximate Version



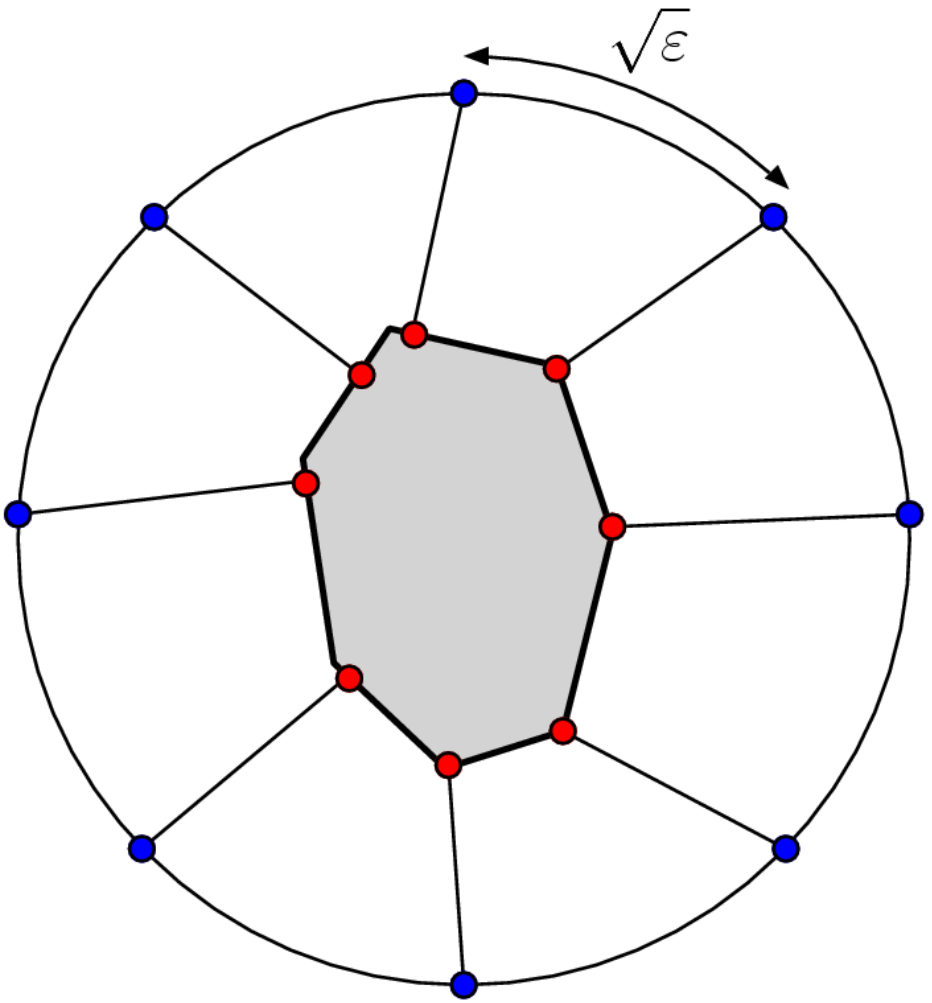
- Assume the polytope has diameter 1
- If the query point is within distance ε of the polytope boundary, either answer is acceptable
- No previous explicit solution to approximate polytope membership
- Polytope approximation is a well studied topic

Bentley et al. (1982) Approximation



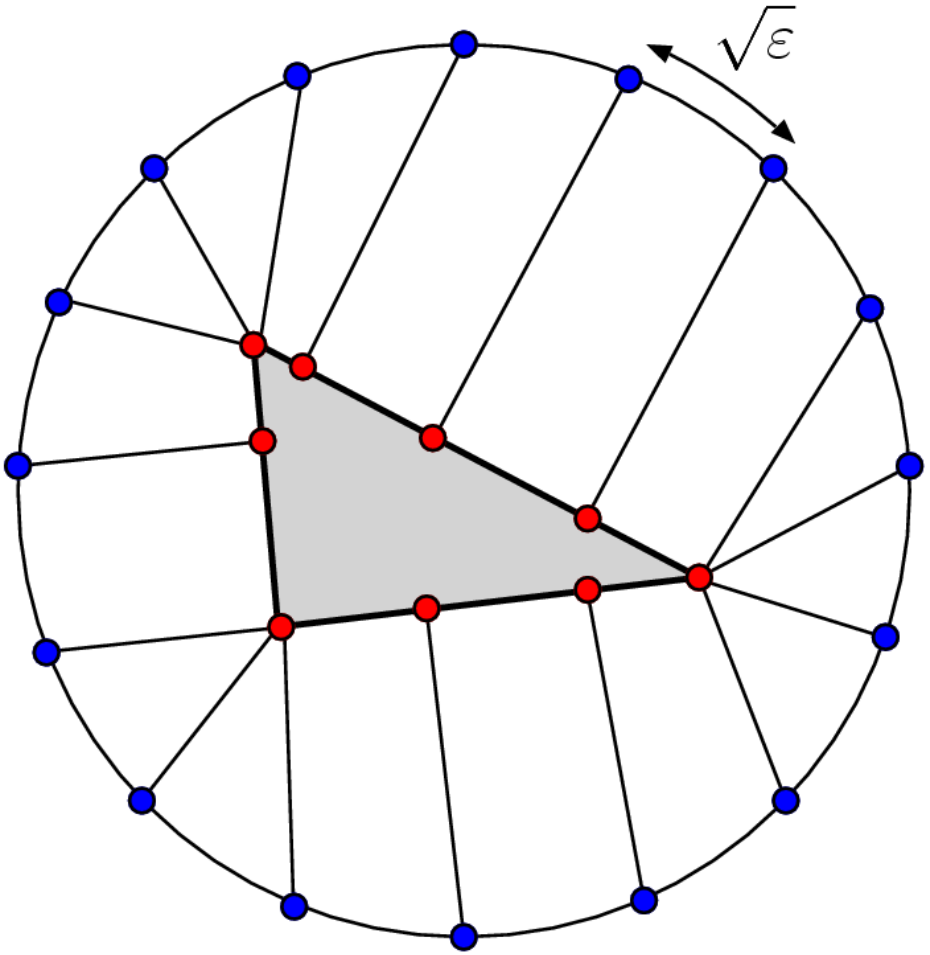
- Create a grid with cells of diameter ε
- For each column, store the topmost and bottomost cells inside the polytope
- A query can be answered by locating the column that contains *q* and comparing with the two extreme values
- Storage: $O(1/\varepsilon^{d-1})$
- Query time: O(1)

Dudley (1974) Approximation



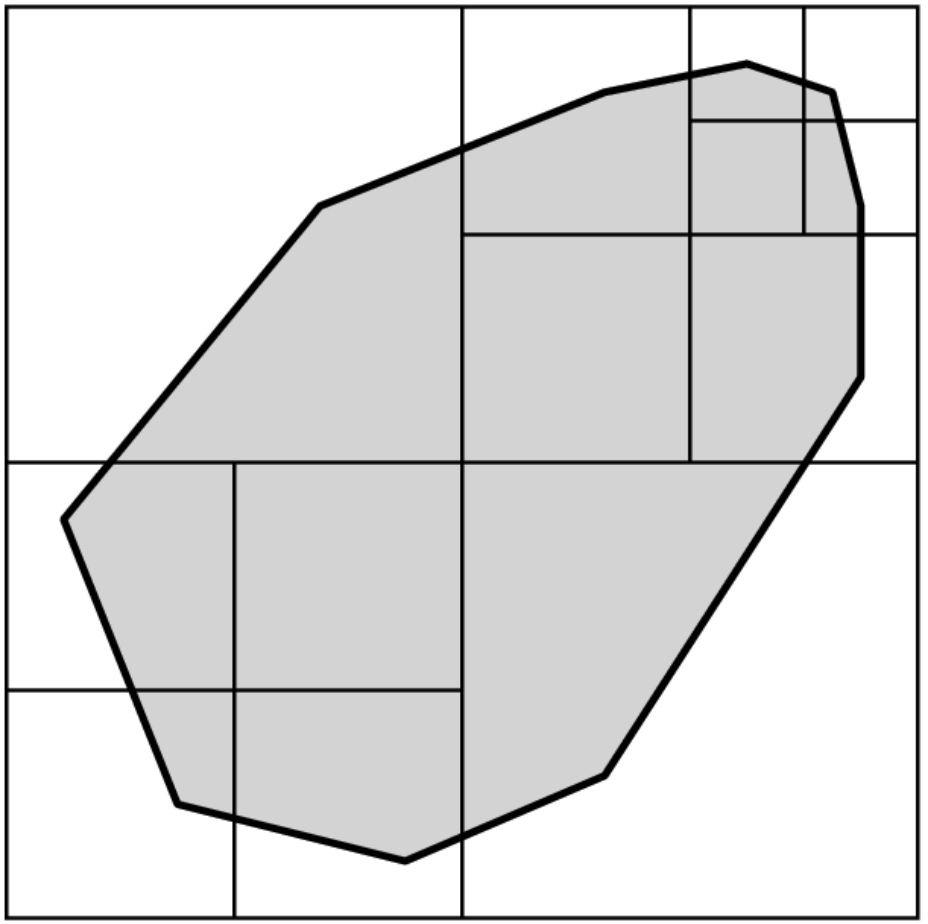
- Every polytope can be approximated with $O(1/\varepsilon^{(d-1)/2})$ halfspaces
- This bound is tight in the worst case (unit ball)
- Inspecting Dudley approximation by brute force we get:
  - Storage: $O(1/\varepsilon^{(d-1)/2})$
  - Query time: $O(1/\varepsilon^{(d-1)/2})$
- Note: See halfspaces as disks of size $\sqrt{\varepsilon}$

Sensitive Approximation



- Some polytopes require much fewer halfspaces than Dudley's bound
- Dudley oversamples regions of very low and very high curvature
- Finding the smallest number of halfspaces reduces to set cover
- A log(1/ε)-approximation can be found efficiently
- See Mitchel and Suri, 1995; Clarkson, 1993

SplitReduce Algorithm



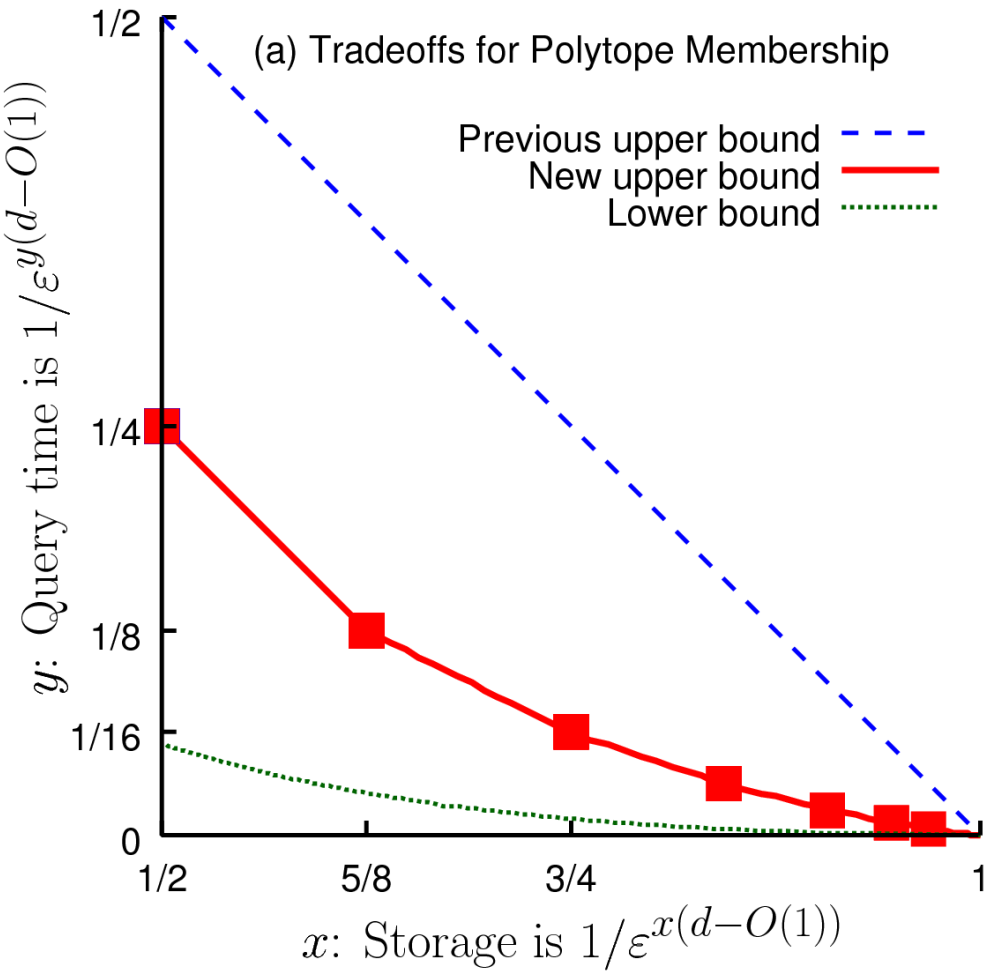
- Find a sensitive approximation to *P*
- Stop if it uses at most *t* halfspaces
- Otherwise, subdivide *P* using a quadtree and recurse for each cell

- Query time: O(*t*) (assuming t > log(1/ε))
- Storage: ???

Analysis for $t \geq 1/\varepsilon^{(d-1)/4}$

- Cells of size $\sqrt{\varepsilon}$ are not subdivided
  - By Dudley, a polytope of diameter $\sqrt{\varepsilon}$ needs only $O(1/\varepsilon^{(d-1)/4})$ halfspaces to be ε-approximated
- The sum of the number of halfspaces in cells of size at least $\sqrt{\varepsilon}$ is $O(1/\varepsilon^{(d-1)/2})$
  - Each Dudley halfspace is only needed within a radius of $\sqrt{\varepsilon}$
- Query time: $O(1/\varepsilon^{(d-1)/4})$
- Storage: $O(1/\varepsilon^{(d-1)/2})$

Tradeoff



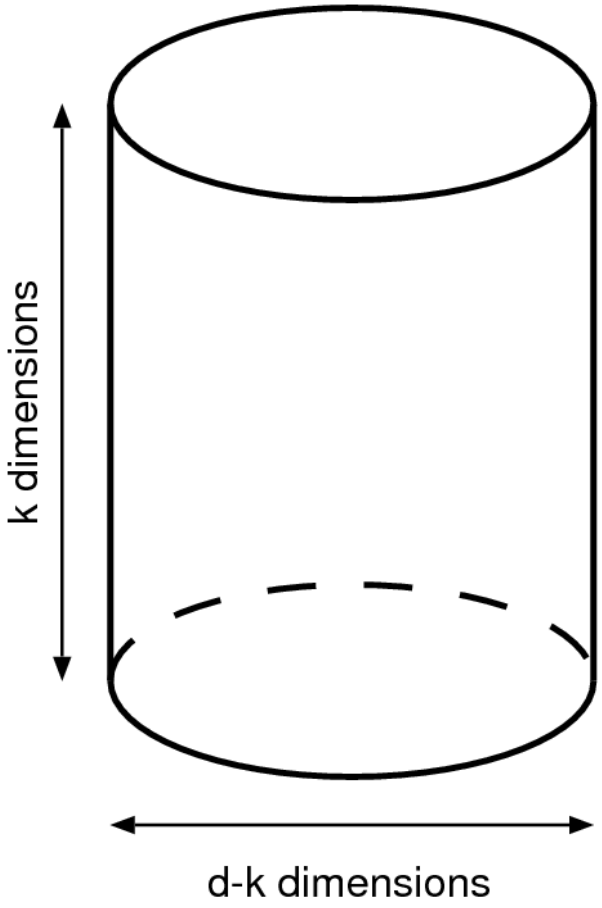
- An inductive application of the previous argument yields a space-time tradeoff
- Query time:

$$t = 1/\varepsilon^{(d-1)/2^k}$$

- Storage:

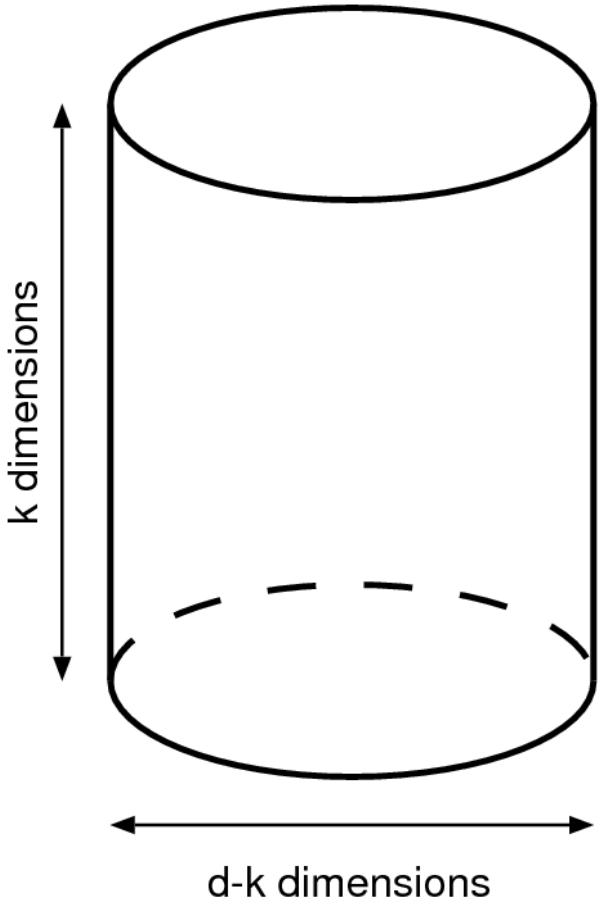
$$O\left(1/\varepsilon^{(d-1)\left(1-\frac{k}{2^k}\right)}\right).$$

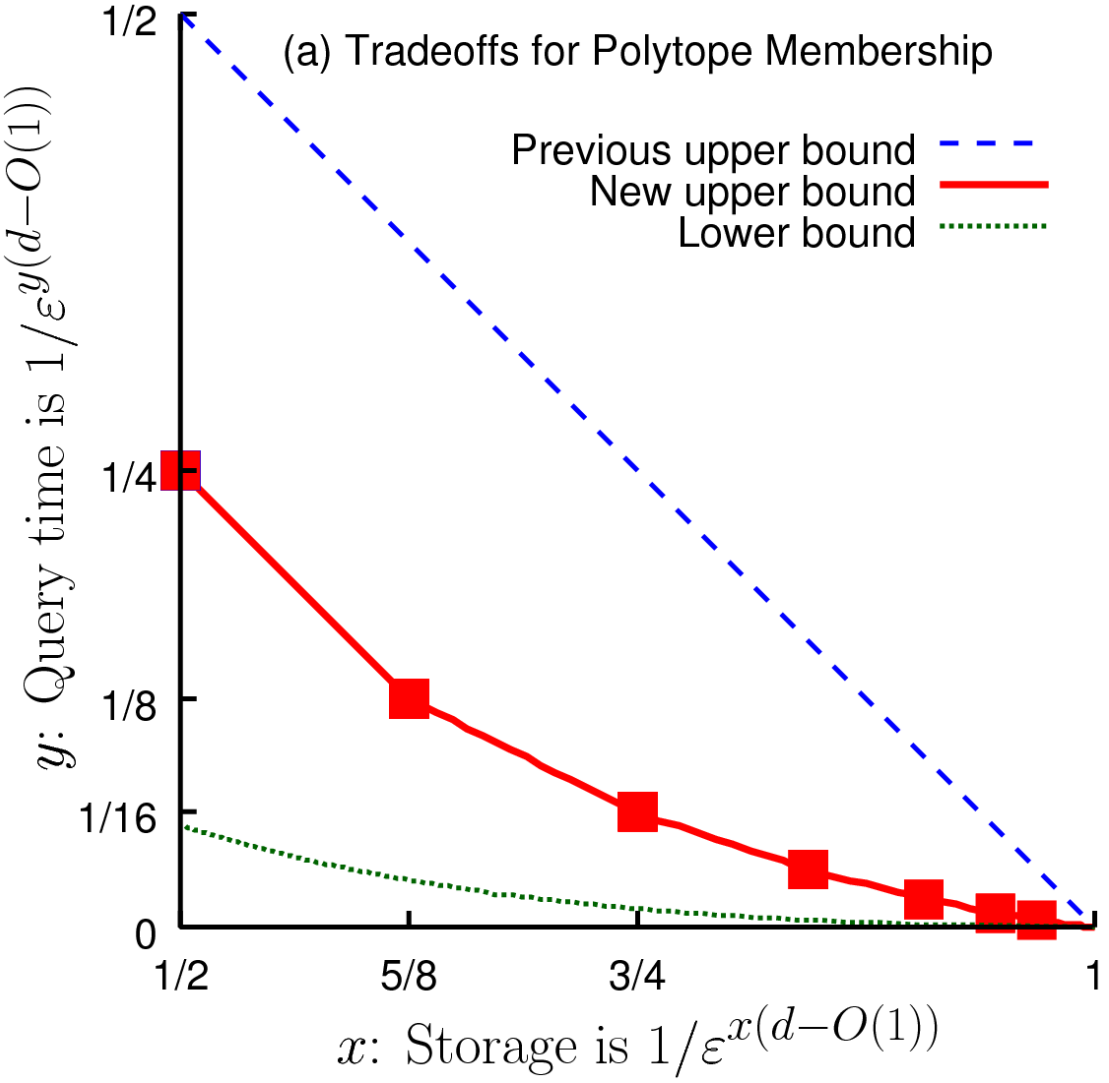
Lower Bound



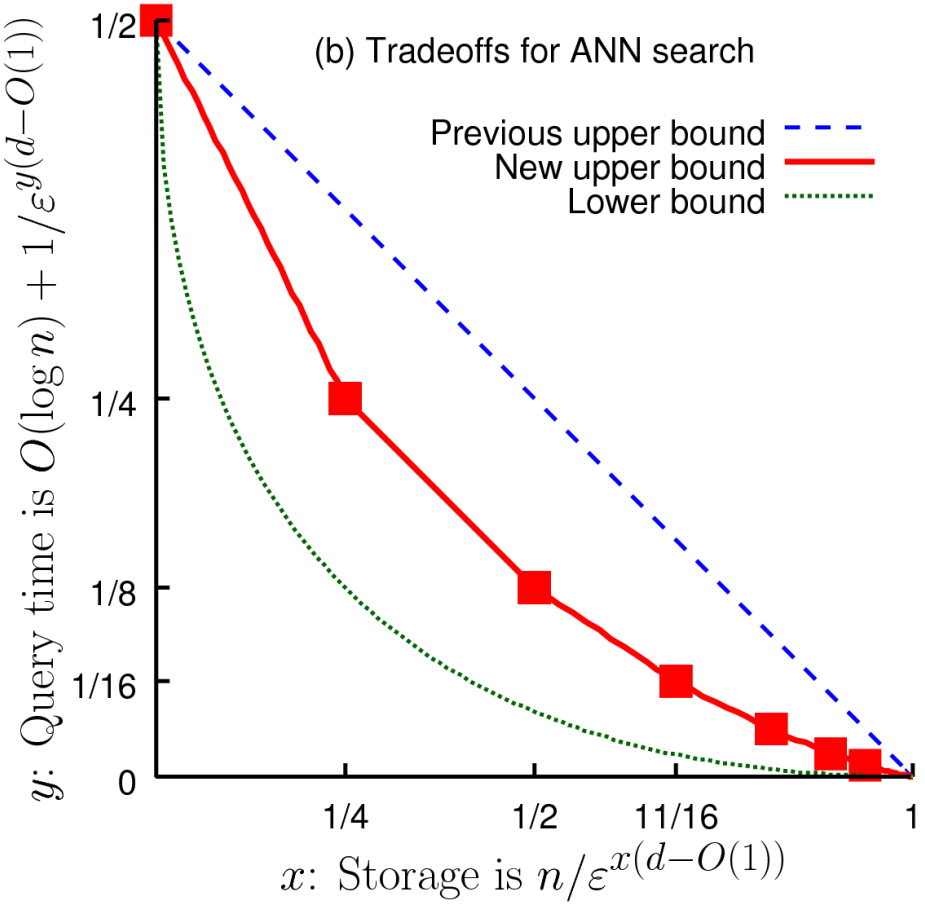
- We also prove a lower bound for the complexity
- The lower bound input polytope is an approximation to a hyper-cylinder
- The cylinder is formed by a ball in (*d*-*k*)-dimensional space extruded in *k* dimensions
- The parameter *k* is chosen to maximize the storage for a given query time

Lower Bound


k dimensions
d-k dimensions

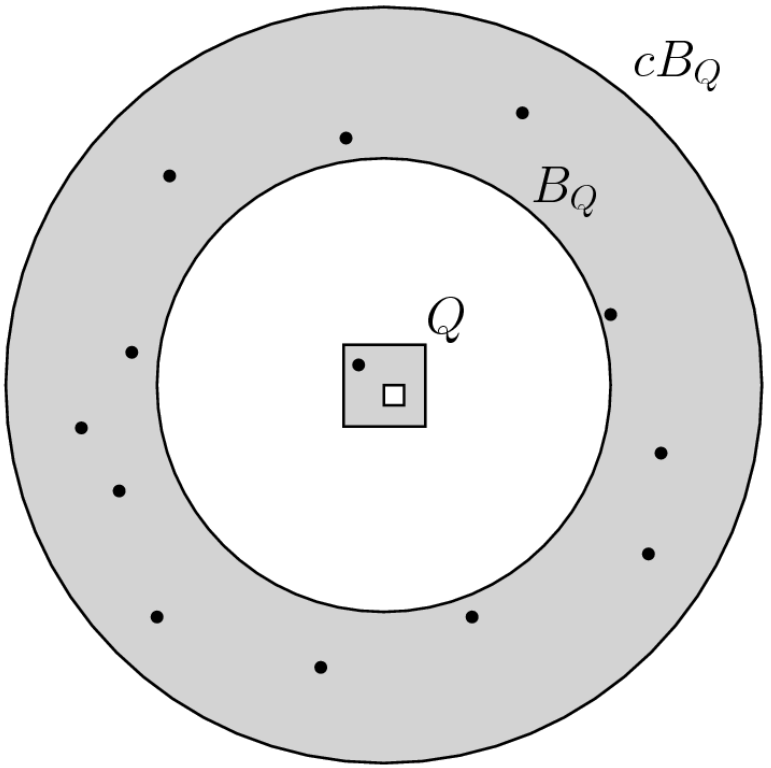

(a) Tradeoffs for Polytope Membership
Previous upper bound
New upper bound
Lower bound
1/2
1/4
1/8
1/16
0
1/2
5/8
3/4
1
y: Query time is $1/\varepsilon^{y(d-O(1))}$
x: Storage is $1/\varepsilon^{x(d-O(1))}$

Approximate Nearest Neighbor (ANN)



- Preprocess *n* points such that, given a query point *q*, we can find a point *p* within distance at most 1+ε times the distance from *q* to its nearest neighbor
- Widely studied
- Optimal solutions for the extremes of the space-time tradeoff (Arya, Malamatos, Mount, 2009)
- We improve the results in the middle

Approximate Voronoi Diagram (AVD)



- Arya et al. show that it is possible to partition space into BBD cells, each associated with candidates to be the ANN for query points in the cell, such that:
  - Total number of candidates: $\tilde{O}(n)$
  - All but 1 candidate are inside a constant annulus
- Using lifting we can reduce the search to log(1/ε) approximate polytope membership queries

Conclusion

- Improved upper bound for approximate polytope membership
- Simple algorithm
- Tight analysis still open
- We also gave a lower bound
- Improves the state of art of ANN searching

Thank you!

Questions?